ON MATHEMATICAL MODELS OF ARTIFICIAL NEURAL NETWORKS

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Abstract. Artificial Neural Networks are in focus. The four-dimensional and the five-dimensional systems are considered. The activation function – hyperbolic tangent is used to model the Artificial Neural Networks. By changing one of the parameters in the system, different types of solutions are obtained: periodic solutions and chaotic solutions. The graphs of all solutions, the dynamics of Lyapunov exponents and 2D and 3D projections of attractors are provided using Wolfram Mathematica.

Keywords: Artificial Neural Networks, periodic solutions, chaotic solutions, Lyapunov exponents.

Introduction

In 1943 two American scientists neurophysiologist and cyberneticianMcCulloch and logician and cognitive psychologistPitts composed the first mathematical model inspired by biological neurons, resulting in the first conception of the artificial neuron, consult the references in [1; 2].

An artificial neural network (ANN) is a computational architecture for processing complex data using multiple interconnected processors and computational paths. Artificial neural networks, created by analogy with the human brain, can train and analyze large and complex data sets that are extremely difficult to process using more linear algorithms [3]. The most active developers of artificial neural networks are Google (Google DeepMind), Microsoft Research, IBM, Facebook (Facebook AI Research), Baidu.

Artificial neural networks are used in various fields: 1) economy and business: forecasting exchange rates, prices for raw materials, assessing the risks of non-repayment of loans, predicting bankruptcies [4]; 2) medicine and healthcare: diagnostics of diseases (artificial neural networks are used for classifying and predicting cancer based on the genetic profile of a given individual), processing of medical images, analysis of the effectiveness of the treatment [1]; 3) robotics: recognition of objects and obstacles in front of the robot, laying the route of movement, control of manipulators, maintaining balance; 4) input and processing of information: recognition of handwritten texts, scanned postal, payment, financial and accounting documents; recognition of speech commands, speech input of text into a computer; 5) agriculture: diagnosing diseases of crops and other plants [4].ANN, implemented in agricultural machinery, will take over the functions that are now performed by a large number of workers. Harvesters controlled from a distance will scan the surrounding area, and the neural network, having studied the received images, will make a decision about watering, weeding, loosening, fertilizing.

Neural networks work with large amounts of data faster and more efficiently than a person. In agriculture, hundreds of hectares are measured, personnel by thousands of employees, and livestock by millions of individuals. This is “big data”. ANN are a tool for solving the most complex problems. It will teach how to save resources, help improve the quality and safety of products, and simplify many processes at all stages of production. The artificial neurons used in artificial neural networks are nonlinear, usually providing continuous outputs, and performing simple functions [1]. An Artificial Neural Network is a mathematical model that tries to simulate the structure and functionalities of biological neural networks. A basic building block of every artificial neural network is an artificial neuron, that is, a simple mathematical model (function) [5].

Activation functions are used to model ANN. The four main functions are the logistic function, hyperbolic tangent, Gaussian function and linear function [1]. We consider models, based on the ordinary differential equations. These equations allow for stable equilibria, stable periodic solutions and chaotic attractors. For low dimensional systems the qualitative and numerical analysis is standard. More problems arise when considering high dimensional systems corresponding to more realistic networks, containing many elements. Our aim is to indicate a passage to investigation of higher order models of ANN. Due to limited space available we restrict ourselves with four and five dimensional systems. We provide examples of systems, which possess periodic solutions and solutions with irregular behavior.

Materials and methods

Our consideration is mostly geometrical. Since four and five dimensional objects are difficult to visualise, we provide two dimensional and three dimensional projections onto appropriate subspaces. Our main intent is to use the 2D and 3D projections of the attractor on different subspaces, to construct the graphs of solutions of the systems and to show the Lyapunov dynamics of the considered systems. In the article for Lyapunov exponents Wolfram Mathematica program "Lynch-DSAM.nb" was used [6].

4D systems

Consider the system

$$\begin{cases} x_1' = \tanh(-x_2 + x_4) - bx_1 \\ x_2' = \tanh(x_1 + x_4) - bx_2 \\ x_3' = \tanh(x_1 + x_2 - x_4) - bx_3 \\ x_4' = \tanh(-x_2 + x_3) - bx_4 \end{cases} \quad (1)$$

The initial conditions are

$$x_1(0) = 1.2; x_2(0) = 0.4; x_3(0) = 1.2; x_4(0) = -1$$

The graph of the system (1) is presented in Fig. 1.

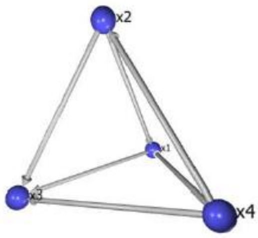

Fig. 1. **Graph, corresponding to the system (1)**

We know two types of attractors in systems (1), namely, stable periodic solutions and chaotic solutions [7]. Consider b = 0.1.

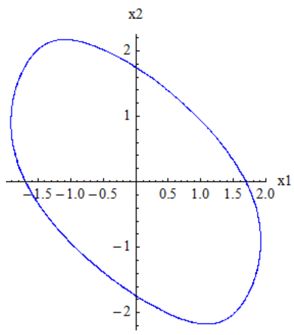

Fig. 2. **Projection of the attractor on 2D subspace ($x_1(t)$, $x_2(t)$)**

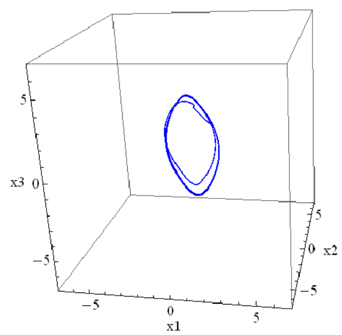

Fig. 3. **Projection of the attractor on 3D subspace ($x_1(t)$, $x_2(t)$, $x_3(t)$)**

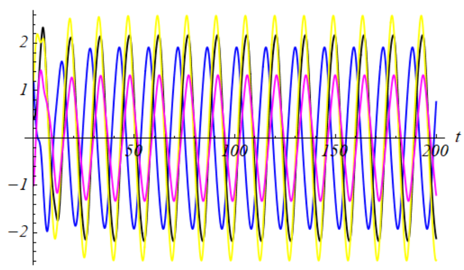

Fig. 4. **Solutions ($x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$) of the system (1)**

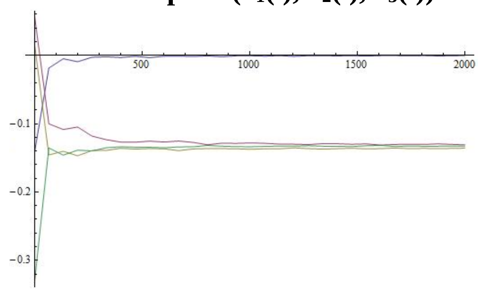

Fig. 5. $\boldsymbol{LE_1 = 0; LE_2 = -0.131; LE_3 = -0.136; LE_4 = -0.133}$

The Lyapunov exponent (LE) is one of the approaches to detect chaos [8]. We have $(0;-;-;-)$, then the system (1) has periodic solutions [9]. Now consider b = 0.045

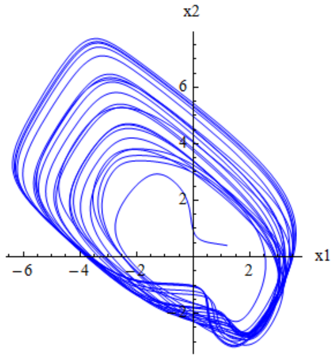

Fig. 6. **Projection of the attractor on 2D subspace ($x_1(t)$, $x_2(t)$)**

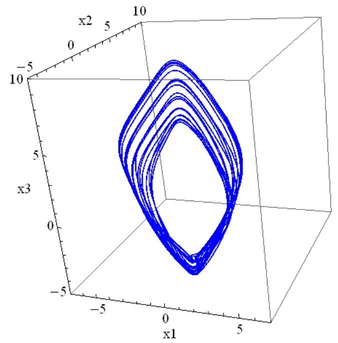

Fig. 7. **Projection of the attractor on 3D subspace ($x_1(t)$, $x_2(t)$, $x_3(t)$)**

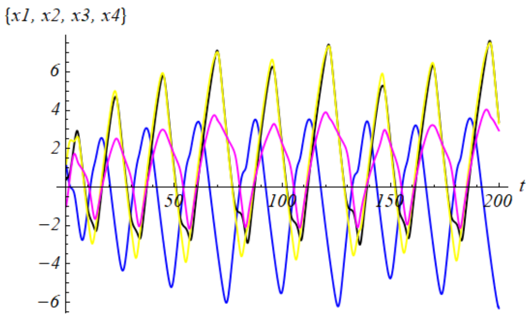

Fig. 8. **Solutions ($x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$) of the system (1)**

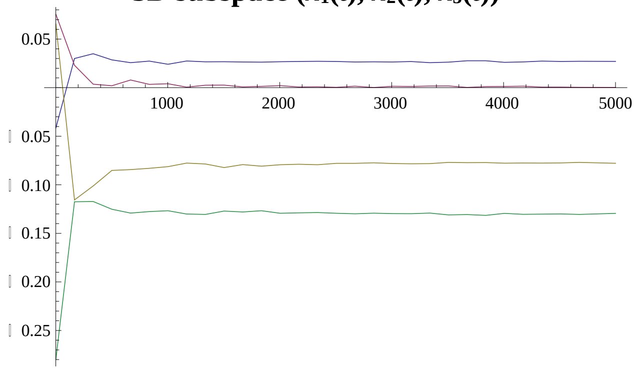

Fig. 9. **LE_1 = 0.03; LE_2 = 0; LE_3 = -0.08; LE_4 = -0.13**

We have (+ ;0;-;-), then the system (1) has chaotic solutions [8].

5D systems

Consider the system

$$\begin{cases} x_1' = \tanh(-x_2 + x_4) - bx_1 \\ x_2' = \tanh(x_1 + x_4) - bx_2 \\ x_3' = \tanh(x_1 + x_2 - x_4) - bx_3 \\ x_4' = \tanh(-x_2 + x_3) - bx_4 \\ x_5' = \tanh(x_1 + x_4 - x_5) - bx_5 \end{cases} \tag{2}$$

The initial conditions are

$$x_1(0) = 1.2; x_2(0) = 0.4; x_3(0) = 1.2; x_4(0) = -1; x_5(0) = -1$$

The graph of the system (2) is presented in Fig. 10.

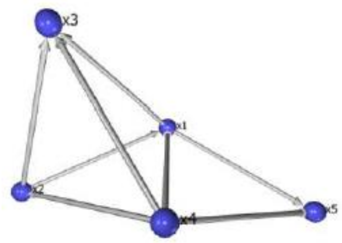

Fig. 10. **Graph, corresponding to the system (2)**

Consider $b = 0.1$.

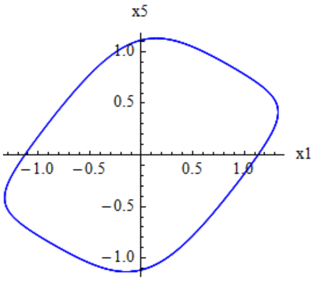

Fig. 11. **Projection of the attractor on 2D subspace ($x_1(t)$, $x_2(t)$)**

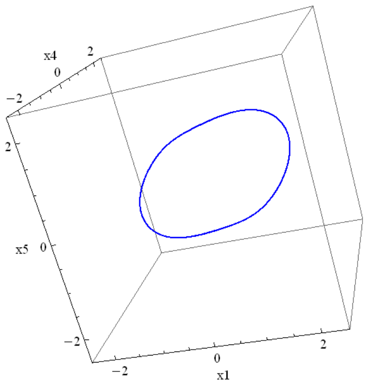

Fig. 12. **Projection of the attractor on3D subspace ($x_1(t)$, $x_4(t)$, $x_5(t)$)**

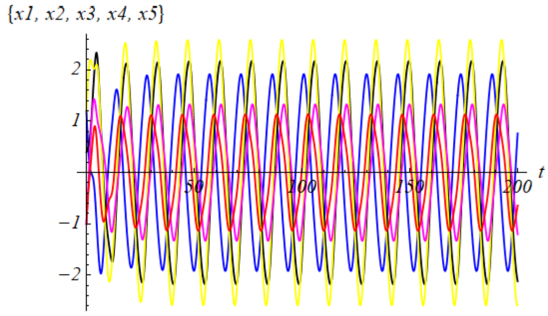

Fig. 13. **Solutions ($x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$, $x_5(t)$) of the system (1)**

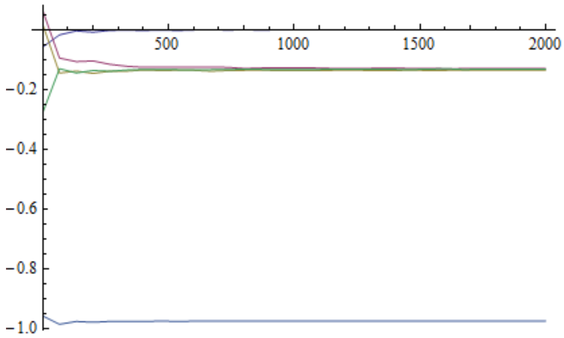

Fig. 14. $LE_1 = 0$; $LE_2 = -0.13$; $LE_3 = -0.14$; $LE_4 = -0.13$; $LE_5 = -0.97$

We have (0;-;-;-;-), then the system (2) has periodic solutions. Consider $b = 0.045$.

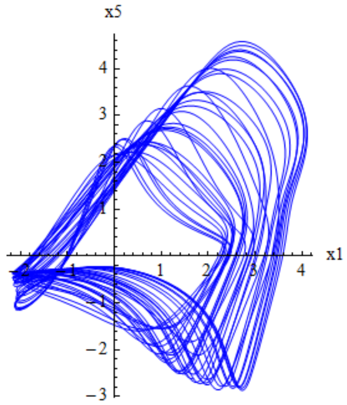

Fig. 15. **Projection of the attractor on2D subspace ($x_1(t)$, $x_5(t)$)**

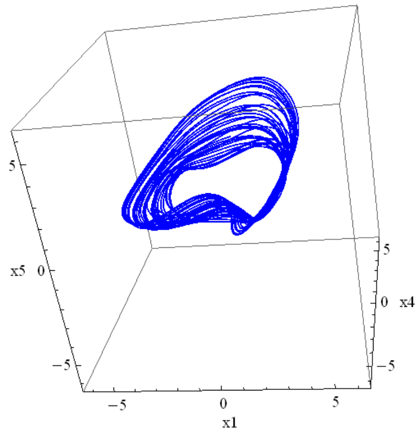

Fig. 16. **Projection of the attractor on 3D subspace ($x_1(t)$, $x_4(t)$, $x_5(t)$)**

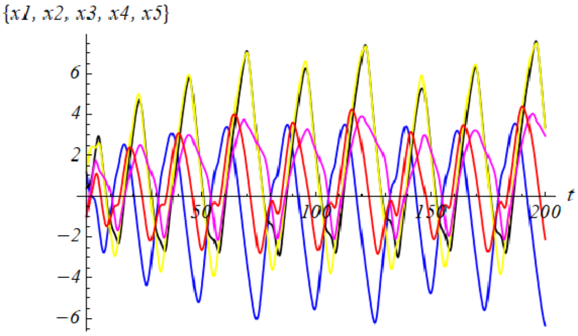

Fig. 17. **Solutions ($x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$, $x_5(t)$) of the system (1)**

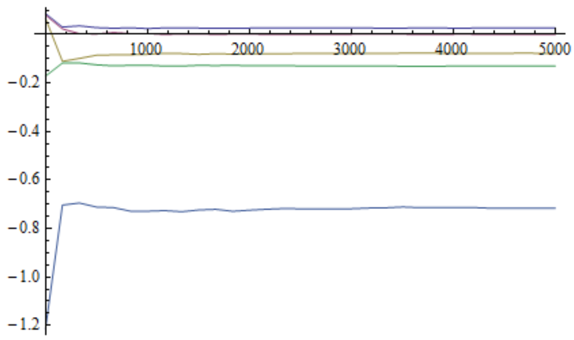

Fig. 18. $LE_1 = 0.03$; $LE_2 = 0$; $LE_3 = -0.08$; $LE_4 = -0.13$; $LE_5 = -0.70$

We have (+ ;0;-;-;-), then the system (2) has chaotic solutions.

Results and discussion

Artificial Neural Networks are mathematical models of networks inspired by biological networks. They are widely used in applications now. The dynamics of ANN can be modeled using dynamical systems represented by continuous ordinary differential equations |(ODE). For brevity, we will call these systems *ANN-type systems*. Dynamical systems used in models of ANN have a specific form and structure, like in (1) and (2), and they consist of two parts, the linear one, and the nonlinear part, which represent the output of an element of ANN after summarizing and processing the inputs from other elements of ANN. This output can be imagined as the response of an element to multiple inputs. Generally, this response is modeled by a sigmoidal function, which are many (Hill's function, logistic function, etc.) In many models the hyperbolic tangent is used. This function is defined on the infinite interval and takes values in the open interval (-1,1). The phase space of ANN systems contains basins of attraction of some special subsets, called attractors. So, in a long run, any trajectory of an ANN-type system goes to one of the attractors. Therefore, the future states of ANN can be predicted if the whole structure of the phase space is known and attractors are detected and located. This is not the case, however, for ANN systems of dimensionality higher than two. So, any knowledge about systems of the form (1) and/or (2) is valuable for the investigation of ANN of the corresponding dimension. In the above sections, an analysis of the behaviour of trajectories to 4D and 5D systems is made. The values of parameters were found (generally there is not a regular procedure for finding them) for both systems, which leads to a sensitive dependence of solutions on the initial data. Graphs of such solutions have irregular forms.

Conclusions

Systems of ordinary differential equations were studied, which are interpreted as artificial neuronal networks. These systems are dependent on multiple parameters. The dynamics of such systems heavily depends upon parameters, and generally any dynamical system can be approximated by systems of the form (1) and (2) at the cost of increasing dimensionality. In this paper four and five-dimensional systems were considered. They have at least one equilibrium, which can be attractive or not. In the second case, a new attractor appears. The behaviour of solutions becomes more and more complicated. Detection of the parameters, ensuring complicated behaviour of trajectories and their sensitive dependence on the initial data, is a challenging problem. In various engineering applications of ANN management and control over a network even in a mathematical model is important. For these purposes, the knowledge of the structure of the phase space and the ability to control connections between elements of a network are the necessary components.

Author contributions

All the authors have contributed equally to creation of this article.

References

[1] da Silva I.N., Spatti D.H., Flauzino R.A., Bartocci Liboni L.H., dos Reis Alves S.F. Artificial Neural Networks: A Practical Course. Springer International Publishing, 2016, 307 p.

[2] Francik S., Ślipek Z., Frączek J., Knapczyk A. Present trends in research on application of artificial neural networks in agricultural engineering. Agricultural Engineering, 2016,Vol. 20, No.4, pp.15-25. ISNN 2083-1587; e - ISNN 2449-5999.

[3] Arjun Baliyan, Kumar Gaurav, Sudhansu Kumar Mishra. A Review of Short Term Load Forecasting using Artificial Neural Network Models. Procedia Computer Science 48, 2015, pp. 121-125.

[4] Abiodun O.I., Jantan A., Omolara A.E., Dada K.V., Mohamed N.A.E., Arshad H. State-of-theart in artificial neural network applications: A survey. Heliyon 4 (2018) e00938. DOI: 10.1016/j.heliyon.2018.e00938

[5] Suzuki K. Artificial Neural Networks. Methodological advances and biomedical applications. InTech Janeza Trdine 9, 51000 Rijeka, Croatia, 2011.

[6] Lynch S. Dynamical Systems with Applications Using Mathematica. Springer, 2017.

[7] Sprott J. C. Elegant Chaos. Algebraically Simple Chaotic Flows. World Scientific Publishing Company, 2010, 302 pages. DOI: 10.1142/7183
[8] Samuilik I., Sadyrbaev F. On a dynamical model of genetic networks. WSEASTRANSACTIONS on BUSINESS and ECONOMICS DOI: 10.37394/23207.2023.20.11
[9] Samuilik I. “Genetic engineering – construction of a network of four dimensions with a chaotic attractor,” Vibroengineering PROCEDIA, Vol.44, pp. 66-70. DOI: 10.21595/vp.2022.22829