MATHEMATICAL MODELLING OF NONLINEAR DYNAMIC SYSTEMS

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Abstract. Attracting sets for systems of ordinary differential equations, which arise in multiple applications, are constructed. The six-dimensional system is in the focus. The construction is based on previously obtained attractors for systems of orders two and three. First, the uncoupled six-dimensional system is considered. Adding some additional elements makes this system coupled. The attractors, however, remain in a modified form. The graphs of all six solutions are provided as visual evidence of the existence of attractors.

Keywords: networks, mathematical model, dynamical system, attractors, chaos.

Introduction

Dynamic systems mean systems of any nature: physical, chemical, biological, societies and populations, ecosystems, as well as financial markets, computing processes, and processes of information transformation [1]. In such systems, unstable and stable foci, cycles, and chaotic behavior of the system are possible [2]. Cycles are widely used in many areas of natural sciences: radio physics, the theory of oscillations, mathematical biology (photosynthesis), chemistry (periodic processes in reactions), aviation (aircraft dead loop), automatic control, mathematical economics, astronomy, medicine (mental illness). Chaotic behavior can be observed in economics. When a crisis occurs, the system loses its dynamic stability and passes to chaos. At the end of time, the economy emerges from the crisis, which means that when the parameter changes, there is a transition from chaos to orderly movement. With economic development, new lines of business emerge from chaotic behavior. Artificial elimination of uncertainty leads to stagnation and degradation of the system since the emergence of progressive directions of development is excluded. As it turned out, the necessary condition for the emergence of chaos in dynamic systems is a dimension of the phase space $n \geq 3$ when the state of the system is characterized by at least three variables.

We consider systems of ordinary differential equations of order greater than three. These systems are of a special form. Let $f_i(z) = [1 + \exp(-\mu_i(z-\theta_i))]^{-1}$. The system of ODE consisting of six equations is

$$\begin{cases} \frac{dx_1}{dt} = f_1(w_{11}x_1 + \cdots + w_{16}x_6) - v_1x_1, \\ \frac{dx_2}{dt} = f_2(w_{21}x_1 + \cdots + w_{26}x_6) - v_2x_2, \\ \quad\cdots\cdots\cdots \\ \frac{dx_6}{dt} = f_6(w_{61}x_1 + \cdots + w_{66}x_6) - v_6x_6. \end{cases} \tag{1}$$

Similar systems of dimensionality two, three, four and arbitrary dimensionality [3] appear in various contexts describing neuronal networks [4; 5], genetic networks [6; 7], telecommunication networks [8], and more. These type models can reflect an evolution in the time t of a network. Network management and control is possible by changing the system parameters. There are a lot of them even in two-dimensional systems.

$$\begin{cases} \frac{dx_1}{dt} = \frac{1}{1+e^{-\mu_1(w_{11}x_1+w_{12}x_2-\theta_1)}} - v_1x_1, \\ \frac{dx_2}{dt} = \frac{1}{1+e^{-\mu_2(w_{21}x_1+w_{22}x_2-\theta_2)}} - v_2x_2. \end{cases} \tag{2}$$

The functions $f_i(z)$ are sigmoidal ones with the characteristic properties 1) monotonically increasing from zero to unity; 2) possessing a unique inflection point. [9; 10] There are many sigmoidal functions, which can be used in models. For instance, Hill's function was used in [6], the logistic function was tried in [4; 5; 11; 12]. The parameters μ_i, θ_i, v_i characterize the system, while the coefficients w_{ij} are elements of the matrix

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$$W = \begin{pmatrix} w_{11} & w_{12} & \dots w_{16} \\ w_{21} & w_{22} & \dots w_{26} \\ \dots & \dots & \dots \dots \\ w_{61} & w_{62} & \dots w_{66} \end{pmatrix}. \quad (3)$$

This matrix contains information about the interaction of nodes x_i: positive w_{ij} means activation of x_i by x_j. Respectively, the negative coefficient means inhibition (repression). Zero entry corresponds to no relation.

The system (1) allows for different interpretations, depending on the field of application. In the theory of genetic regulatory networks (GRN) it is understood as a description of possible scenarios of network evolution. These scenarios depend on a set of parameters, on the matrix W; on initial conditions. The influence of all these factors is summarized in the description and configuration of the attractors of the system (1). In [6] (see also [13]) the solution vector $X(t) = (x_1(t), \dots, x_6(t))$ is treated as the current state of a network at a time moment t. Future states are dependent on the topology of the phase space. If $X(t)$ is the basis of attraction of a particular attractor, it eventually tends to one. We will consider attracting sets of the system (1). We construct them by combining attractors that appear in low-dimensional systems (2D and 3D).

Materials and methods

Our consideration is geometrical. All processes of interest to us take place in a bounded parallelepiped G and our main intent is to use the 3D projections of the attractor on different subspaces, to construct the graphs of solutions for understanding and managing the system. Visualizations, where possible, are provided. Computations are performed using Wolfram Mathematics.

2D systems

We know two types of attractors in systems (2), namely, stable equilibria and stable periodic solutions. Stable critical points easily can be obtained considering activation or inhibitory cases. Periodic solutions appear in systems (2) with the regulatory matrices of the form

$$W = \begin{pmatrix} k & 2 \\ -2 & k \end{pmatrix} \quad (4)$$

for appropriate positive values of k. Examples can be found in [7].

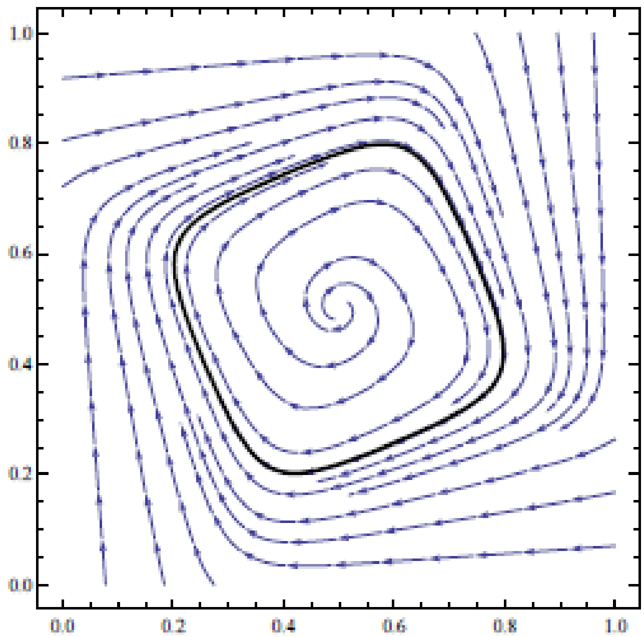

Fig. 1. **Periodic solution, $\mu_1 = \mu_2 = 10$, $\theta_1 = 1.5$, θ_2 =-0.5**

3D systems

We know that three dimensional systems

$$\begin{cases} \frac{dx_1}{dt} = \frac{1}{1+e^{-\mu_1(w_{11}x_1+w_{12}x_2+w_{13}x_1-\theta_1)}} - v_1x_1, \\ \frac{dx_2}{dt} = \frac{1}{1+e^{-\mu_2(w_{21}x_1+w_{22}x_2+w_{23}x_3-\theta_2)}} - v_2x_2, \\ \frac{dx_3}{dt} = \frac{1}{1+e^{-\mu_3(w_{31}x_1+w_{32}x_2+w_{33}x_3-\theta_3)}} - v_3x_3 \end{cases} \tag{5}$$

can have stable equilibria, stable periodic solutions [3], and a chaotic attractor [4; 5].

6D systems

Consider the six-dimensional system (1). Let the regulatory matrix be

$$W = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & -2 & 1 \end{pmatrix}. \tag{6}$$

This system consists of three independent two-dimensional systems, which have an attractor depicted in Fig. 1. The resulting attractor is a product of three two-dimensional ones and is, therefore, periodic. A trial solution with the initial values

$$x_1(0) = 0.68, x_2(0) = 0.3, x_3(0) = 0.1, x_4(0) = 0.6, x_5(0) = 0.2, x_6(0) = 0.1 \tag{7}$$

was used to reveal the six-dimensional attractor.

Change now two elements at the right upper (w_{16}) and left lower (w_{61}) corners. Let $w_{16} = w_{61} = 0.5$. The six-dimensional system (1) is already coupled. The trial solution still tends to be a periodic attractor (a different one), however. The graphs of all six solutions $x_i(t)$ are depicted in Fig. 2 and Fig. 3. Other parameters are

$$\mu_i = 10, i = 1, \ldots, 6;\ \theta_i = 1.2, i = 1,3,5;\ \theta_j = -0.6, j = 2,4,6.$$

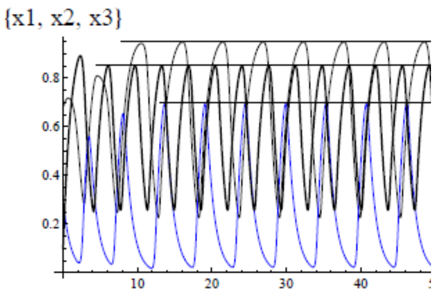

Fig. 2. **Graphs of $x_i(t), i = 1, 2, 3$**

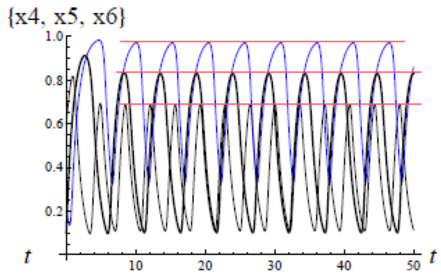

Fig. 3. **Graphs of $x_i(t), i = 4, 5, 6$**

6D system from 3D system

Our intent now is to create a six-dimensional attractor from three-dimensional ones. For this, we take the three-dimensional system (5) with the following set of parameters

$$\begin{gathered} v_1 = 0.65, v_2 = 0.42, v_3 = 0.1; \\ \mu_1 = 7, \mu_2 = 7, \mu_3 = 13; \\ w_{11} = 0, w_{12} = 1, w_{13} = -5.64; \\ w_{21} = 1, w_{22} = 0, w_{23} = 0.1; \\ w_{31} = 1, w_{32} = 0.02, w_{33} = 0; \\ \theta_1 = 0.5, \theta_2 = 0.3, \theta_3 = 0.7. \end{gathered} \tag{8}$$

The respective three-dimensional system was studied in [4; 5]. The authors of [4; 5] claim that this system has a chaotic attractor. It is depicted in Fig. 4. The irregular behavior of three solutions can be seen in Fig. 5.

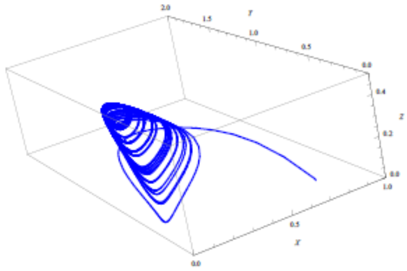

Fig. 4. **3D chaotic attractor**

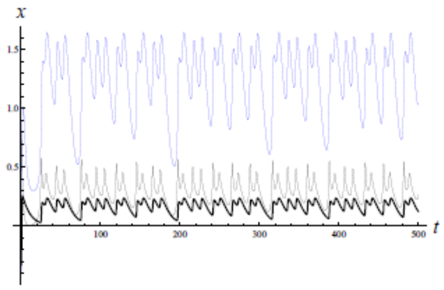

Fig. 5. **Graphs of $x_i(t), i = 1, 2, 3$**

Consider a six-dimensional system with the regulatory matrix

$$W = \begin{pmatrix} 0 & 1 & -5.64 & 0 & 0 & 0 \\ 1 & 1 & 0.1 & 0 & 0 & 0 \\ 1 & 0.02 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -5.64 \\ 0 & 0 & 0 & 1 & 0 & 0.1 \\ 0.5 & 0 & 0 & 1 & 0.02 & 0 \end{pmatrix}. \qquad (9)$$

It would be uncoupled if the element w_{61}be zero. Then we would have a six-dimensional chaotic attractor which is the product of two identical three-dimensional attractors as in Fig. 4. But w_{61} is set to 0.5. The six-dimensional system is coupled now. The new chaotic attractor exists and some three-dimensional projections are depicted in Fig. 6 and Fig. 7.

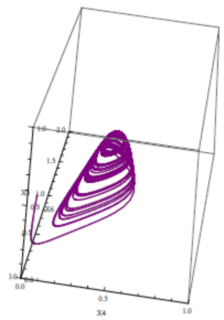

Fig. 6. **Projections of 3D chaotic attractor for the case (9) (*x*4; *x*5; *x*6)**

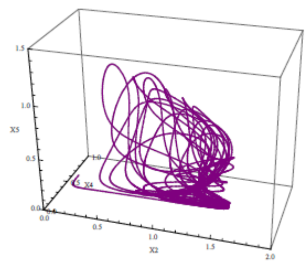

Fig. 7. **Projections of the attractor on (*x*2; *x*4; *x*5)**

The solutions for the system (1) with the matrix (9) are depicted in Fig. 8 and Fig. 9. The solutions have irregular form. They are different in Fig. 8 and Fig. 9 because of the non-zero element w_{16}.

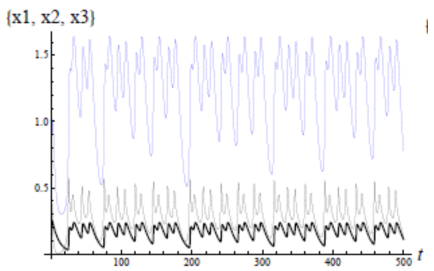

Fig. 8. **Solutions $x_1(t), x_2(t), x_3(t)$**

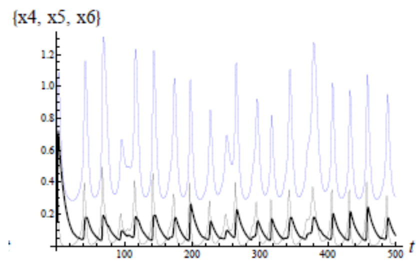

Fig. 9. **Solutions $x_4(t), x_5(t), x_6(t)$**

Consider the six-dimensional system with parameters

$$v_1 = v_2 = v_4 = v_5 = 0.5, v_3 = v_6 = 1;$$

$$\mu_1 = \mu_2 = \mu_4 = \mu_5 = 7, \mu_3 = \mu_6 = 12;$$
$$\theta_1 = 0.5, \theta_2 = 0.3, \theta_3 = 0.7, \theta_4 = 0.5, \theta_5 = 0.3, \theta_6 = 0.7$$

and with the regulatory matrix

$$W = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 1.2 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0.1 & 0 \end{pmatrix}$$

The respective results are shown in Figures 10 – 15.

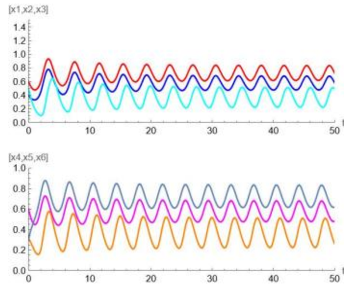

Fig. 10. **Solutions $x_i(t)$ for the case $w_{16} = 0$**

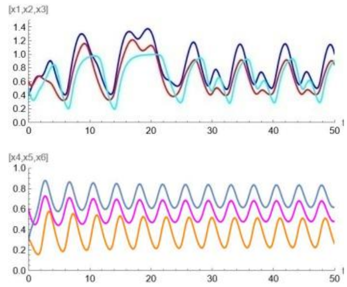

Fig. 11. **Solutions $x_i(t)$ for the case $w_{16} = 1.2$**

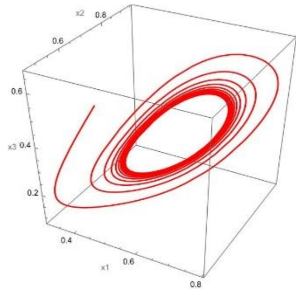

Fig. 12. **Projections of the attractor on (x1; x2; x3) for the case $w_{16} = 0$**

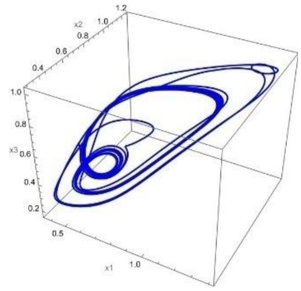

Fig. 13. **Projections of the attractor on (x1; x2; x3) for the case $w_{16} = 1.2$**

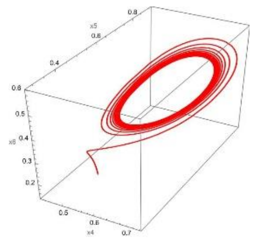

Fig. 14. **Projections of the attractor on (x4; x5; x6) for the case $w_{16} = 0$**

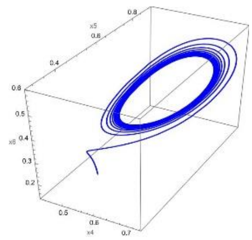

Fig. 15. **Projections of the attractor on (x4; x5; x6) for the case $w_{16} = 1.2$**

Results and discussion

Systems of the form (1) are used in modeling diverse complex networks, such as neuronal networks, genetic networks, and telecommunication networks. The interrelation between nodes in genetic networks is described by a regulatory matrix W. Other adjustable parameters can be used to manage and control a network. To trace the evolution of a network one has to consider attractors in a mathematical model. Attractors can be studied numerically and occasionally analytically. It is important to study the structure of attractors, their locations, and their dependence on the parameters of the system. For high-dimensional systems, the existence of unknown attractors is expected. The role of these attractors in real genetic networks seems to be the most intriguing problem. In this article we have studied six-dimensional systems, using elements of reverse engineering, that is, constructing a network with a stable periodic attractor. The low-dimensional systems, which were studied before, were used to construct a system of dimension six. An attractor for this system is generated by attractors of low-dimensional ones. These attractors can be studied numerically and in some cases analytically. New attractors can be obtained, using this approach. The system obtained has a block structure. Further investigation can be made by filling zero fields with non-zero elements. This is possible without any restrictions on the dimensions of the involved systems. The new system is generally structurally stable. For large perturbations structurally new attractors can appear as well as chaotic behavior. Combinations of attractors of low-dimensional systems are arbitrary and have no limits. This approach can be used for problems of reverse engineering also. New networks with prescribed properties can be constructed, combining one big network of multiple subnetworks with known attractors. This is a perspective plan for further study in this direction. Finally, it should be mentioned that the networks, described by systems of the form (1), are common in other areas. For instance, telecommunication networks of different kinds can be managed by using schemes, first tested on the respective mathematical model. For this, knowledge of substantial properties of mathematical objects, like system (1), is of great importance. Future states of networks described by systems of the form (1), can be calculated if the attractors and locations of the initial states are known. Attractors for high-dimensional systems can be constructed, based on the knowledge of attractors for systems of lower dimensions. The attractors, obtained by perturbations of the regulatory matrices W, can have similar attractors. Combinations of lower dimension attractors of various types are possible and should be studied.

Conclusions

1. Two- and three-dimensional systems of ordinary differential equations, modeling genetic networks, can have stable periodic solutions.
2. A six-dimensional system, modeling genetic networks of the same size, can be constructed as an uncoupled system containing two or three-dimensional blocks.
3. Attractors of six-dimensional systems can be obtained as products of attractors of lower dimensional systems.
4. This method can be used for engineering purposes of constructing artificial genetic like networks with given properties.
5. Perturbation of systems, obtained by this method, can produce new systems with irregular behavior of solutions. Studying these systems may shed light on the mechanism of creation of chaotic attractors.

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Author contributions

All the authors have contributed equally to creation of this article.

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